Surface integrals:

need parametrization.

$$\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$$

$$\Phi(D) = S$$

2 tangent vectors $T_u = \frac{\partial \Phi}{\partial u}$, $T_v = \frac{\partial \Phi}{\partial v}$

<u>surface integral of function:</u>

$$\boxed{\iint_S f\, dS = \iint_D f(\Phi(u,v))\, \|T_u \times T_v\|\, du\, dv}$$

$$\text{area}(S) = \iint_S 1\, dS$$

Surface integral of vector field:

$$\boxed{\iint_S F \cdot dS = \iint_D F(\Phi(u,v)) \cdot (T_u \times T_v)\, du\, dv}$$

⚠ depends on orientation

i.e. one of the two sides of S has to be specified as the positive side

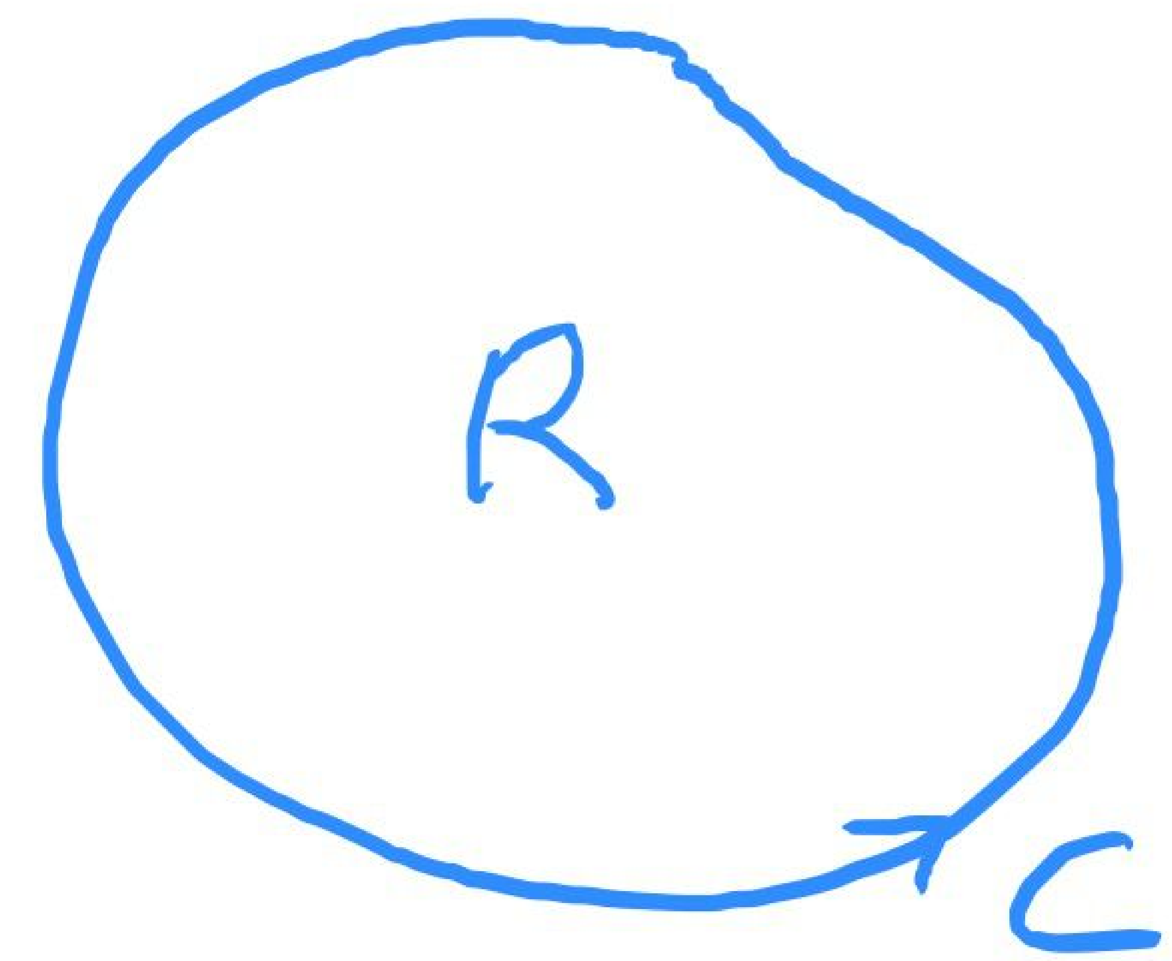
$T_u \times T_v$ has to point to positive side.

if it points to other side $\Rightarrow$ have to put minus sign in front of integral

e.g. if S graph of function

$\rightarrow$ $T_u \times T_v$ points to upper side if z-coordinate > 0.

Green's Theorem



C closed curve
which encircles region R
orientation counter clockwise.

$F : \mathbb{R}^2 \to \mathbb{R}^2$ vector field

$$F(x,y) = (P(x,y), Q(x,y))$$

$$\Rightarrow \boxed{\int_C F \cdot ds = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy}$$

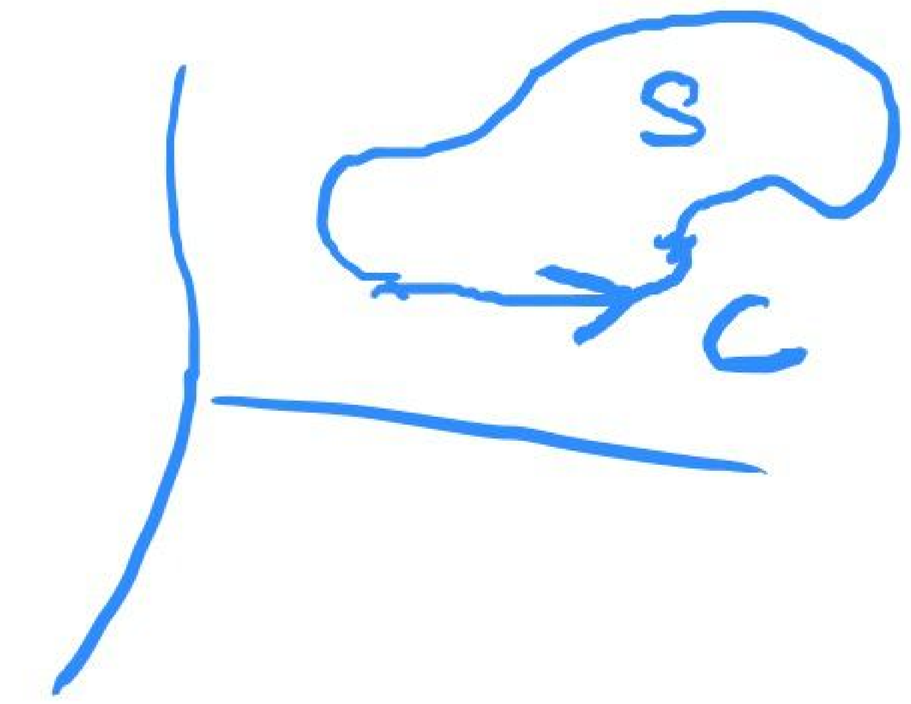
important special case: $F(x,y) = (-y, x)$

$$\text{area}(R) = \frac{1}{2} \int (-y, x) \cdot ds$$

$$= \frac{1}{2} \int -y\,dx + x\,dy$$

Stokes' Theorem

generalization of Green's theorem to surfaces S in $\mathbb{R}^3$ which are bounded by curve C.



$$\boxed{\int_C F \cdot ds = \iint_S \operatorname{curl} F \cdot dS}$$

- use if you are asked to calculate $\iint_S \underline{\operatorname{curl}} F \cdot dS$
- could be useful to calculate $\int_C F \cdot ds$ if $\operatorname{curl} F$ is of a nice form.

Gauss Divergence Theorem

involves $W \subset \mathbb{R}^3$ and its boundary ∂W
(= surface)

$$\iint_{\partial W} F \cdot dS = \iiint_W \operatorname{div} F \, dV$$

orientation: pos. side = outside.

- can be useful for calculating surface integrals where S = boundary of some $W \subset \mathbb{R}^3$

Example: Calculate $\iint_S F \cdot dS$ for S = sphere of radius 2

$F(x,y,z) = (x+yz, -y+xz, z+xy)$

here: $\operatorname{div} F = \partial/\partial x(x+yz) + \partial/\partial y(-y+xz) + \partial/\partial z(z+xy)$
$= 1 - 1 + 1 = 1$

$$\Rightarrow \iint_S F \cdot dS = \iiint_B 1 \, dx\,dy\,dz = \text{volume}(B) = \frac{4}{3}\pi 2^3 = \boxed{\frac{32}{3}\pi}$$

posted: 2 practice final

Some of the problems of first practice final done last class ($\rightarrow$ lecture notes)

Example for Stokes' Theorem

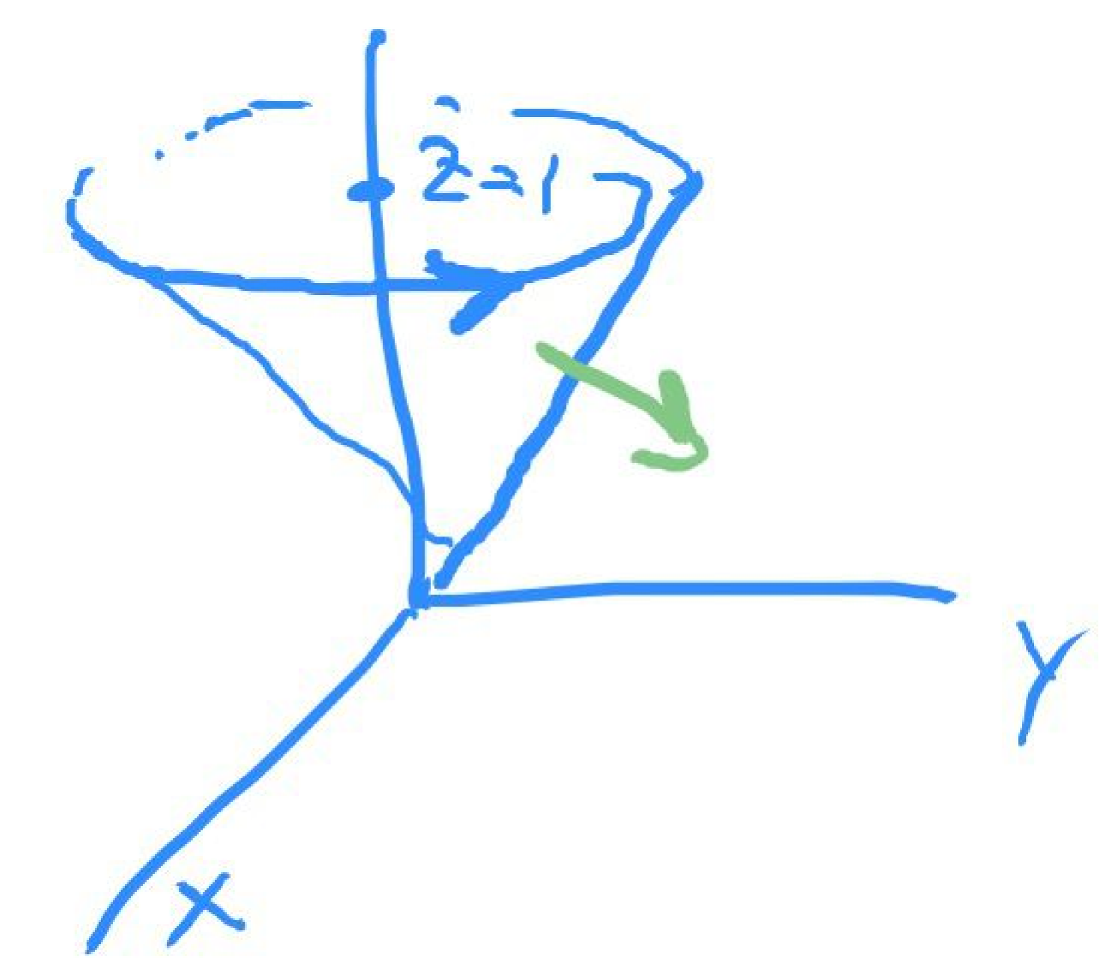
Calculate $\iint_S \text{curl}\, F \cdot dS$ $\qquad$ $\text{curl}\, F = \nabla \times F$

where $\boxed{F(x,y,z) = (-z^2 y,\ z^2 x,\ z^4)}$

$S =$ cone $z = \sqrt{x^2+y^2}$

with $0 \le z \le 1$

normal vector pointing outwards:



Sol: Use Stokes' theorem:

boundary C of S = $(x,y,1)$, $x^2+y^2=1$

parametrize: $\boxed{c(t) = (\cos t, \sin t, 1)}$ $\quad$ ⚠ wrong orientation!

$$\Rightarrow \iint_S \operatorname{curl} F \cdot dS = -\int_0^{2\pi} F(\cos t, \sin t, 1) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= -\int_0^{2\pi} (-\sin t, \cos t, 1) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= -\int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \boxed{-2\pi}$$